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Bayesian Estimation for Exponentiated Inverted Weibull Distribution under Progressively Type-II Censoring Scheme with Binomial Removals

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Abstract The present study pertains to the estimation problem for the parameters and reliability characteristics of the exponentiated inverted Weibull (EIW) distribution under progressively type-II censoring scheme, where the units/ items are removed randomly during the experiments. The number of units/ items removed at each failure follows a Binomial probability law. For this purpose, maximum likelihood and Bayes estimators of exponentiated inverted Weibull distribution have been obtained. Bayesian procedure has been considered under the symmetric and asymmetric loss functions, while the model parameter follows the gamma prior distribution. Furthermore, a Monte Carlo simulation study is carried out to compare the performance of proposed estimators with corresponding maximum likelihood estimators (MLEs) in terms of their simulated risks.

Keywords Bayes estimator; LINEX loss function; Maximum likelihood; MCMC Method; Reliability function; Squared error loss function; Progressive type-II censoring scheme; Hazard function.

1. Introduction

The two-parameter exponentiated inverted Weibull (EIW) distribution has been proposed by Flaih et. al. [1]. The probability density function (pdf) of this model is given by,

$$f(x|\theta,\beta) = \theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}; \qquad x>0; \theta>0, \beta>0 \tag{1}$$

and distribution function is of the form

$$F(x|\theta,\beta) = \left(e^{-x^{-\beta}}\right)^{\theta}; \qquad x>0 \tag{2}$$

Here, θ and β both are shape parameters. The reliability (i.e., the probability of failure after time t) and hazard function (instantaneous failure rate) for model (1) with two shape parameters θ and β are given by,

$$R(t|\theta,\beta) = 1-\left(e^{-t^{-\beta}}\right)^{\theta}; \qquad t>0 \tag{3}$$

and

$$H(t|\theta,\beta) = \frac{\theta\beta t^{-(\beta+1)}\left(e^{-t^{-\beta}}\right)^{\theta}}{1-\left(e^{-t^{-\beta}}\right)^{\theta}}; \qquad t>0 \tag{4}$$

It may note here that the above equation reduces to the standard inverted Weibull (IW) distribution for $\theta=1$ and also for the second parameter $\beta=1$, it represents the exponentiated standard inverted exponential distribution. Hence, the exponentiated inverted Weibull (EIW) distribution is nothing but a generalization of the exponentiated inverted exponential distribution as well as the inverted Weibull distribution. This new distribution also have some physical importance, as if there are m-components in a parallel system and the lifetimes of these components are independents and identically distributed ($i.i.d.$) as exponentiated inverted Weibull distribution. Then, the system lifetime variable has also exponentiated inverted Weibull distribution.

In lifetime scenario, it is often difficult to observe all the units/items of the experiment due to some specific or unspecific reasons like time and cost constraints. So, in this situation, we can remove some units from the experiment while they are still alive i.e, one can call it as censored data. In survival and reliability studies, we usually deal with these censored data. The type-I and type-II are the two most common and popular censoring schemes which is widely used in the fields of survival and reliability studies. In type-I censoring schemes, the experimental time is fixed but the number of observed failure is a random variable while in type-II censoring schemes, number of observed failure is fixed but the experimental time is a random variable, but none of these censoring schemes have discussed the importance of removals of the units/items occurs during the experiment due to some other uncontrolled causes. To overcome from this difficulty, a new censoring schemes is introduced besides the above two schemes, which is progressively type-I and progressively type-II censoring schemes, which allow the removals of the experimental units during the experiment. Here in this paper, we emphasise only on progressive type-II censoring scheme, which may be describes as follows: Suppose, we have n experimental units are put on test at time 0 and going to observe m failure units during the experiment. The experiment proceeds in such a way that when first failure X_1 is observed, R_1 of the surviving units are randomly selected from remaining $(n-1)$ surviving units and then removed i.e, we get R_1 removals from the experiment. And immediate after the second failure X_2 is obtained, again, R_2 of the surviving units are randomly selected from remaining $(n-R_1-2)$ surviving units and removed i.e, R_2 removals obtained. This procedure continues untill the m^{th} failures obtained. Then, at this instance, the experiment terminates and remaining $R_m = n - R_1 - R_2 - R_3 - \ldots - R_{m-1} - m$ surviving units are randomly removed from the experiment. If theses removals $R_1 = R_2 = R_3 = \ldots R_{m-1} = R_m = 0$, then $m = n$, which correspond to complete sample situation and if $R_1 = R_2 = R_3 \ldots = R_{m-1} = 0$, then $R_m = n - m$, which is simply conventional type-II censoring scheme. Thus, the progressive type-II censoring scheme is the generalization of type-II censoring schemes. Statistical inferences based on estimation of parameters for different lifetime models under progressive type-II censoring scheme have been studied by several authors such as Cohen [7], Mann [9], Child and Balakrishnan [6], and Balakrishnan and Aggrawala [3] and so on. Note that in these schemes, the removals $R_1, R_2, R_3, \ldots$, are pre-fixed. However, in some practical situations, the number of removals may occur at random, see, Tse et. al. [15], Wu and Chang [16], Yuen and Tse [17] etc. For example, Consider a doctor perform an experiment with n cancer patients but after the death of the first patient, some patient leave the experiment and go for treatment to other doctor/hospital. Similarly, after the second death a few more leave and so on. Finally, the doctor stops taking observation as soon as the predetermined numbers of deaths (say, m) are recorded.

An important element, in point estimation problem, is the specification of the loss function. The most popular loss function used in the estimation problem is the quadratic or squared error loss function (SELF), which can be easily justified on the grounds of minimum variance-unbiased estimation. However, the weakness of this loss function is that it is symmetric and gives an equal weight to the overestimation and underestimation of the same magnitude. But, in some real situations, overestimation can lead to more severe or less severe consequences than underestimation, or vice versa. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. Subsequently, the use of an asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameters. Also, use of symmetric loss function may be inappropriate as has been recognized by Canfield [4] and Varian [18]. Thus, a number of asymmetric loss functions are available in the statistical literature, and one of the most widely used asymmetric loss function is the LINEX (linear - exponential) loss function, originally proposed by Varian [18] and popularized by Zellner [19], which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa. Let,

$$\Delta = (\hat{\theta} - \theta)$$

Where $\hat{\theta}$ is an estimate of θ. LINEX loss function may be expressed as,

$$L(\Delta) \propto \left(e^{\delta\Delta} - \delta\Delta - 1\right); \qquad \delta \neq 0 \tag{5}$$

Where δ is the loss parameter which reflects the direction and degree of asymmetry. The loss parameter δ allows different shapes of this loss function. If $\delta > 0$, then the LINEX loss function is quite asymmetric about zero with overestimation being more costly than underestimation and vice-versa. For δ close to zero, the LINEX loss function is approximately squared error loss and therefore almost symmetric. Several authors have used this loss function in various estimation and prediction problems. Using this loss function, the posterior expectation of the LINEX loss function (5) is,

$$E_\theta\left[L(\hat{\theta}-\theta)\right] \propto e^{\delta\hat{\theta}} E_\theta\left[e^{-\delta\theta}\right] - \delta\left(\hat{\theta} - E_\theta[\theta]\right) - 1 \quad (6)$$

Where, $E_\pi(\cdot)$ denotes the posterior expectations with respect to posterior density of θ. By a result of Zellner [19], the (unique) Bayes estimator of parameterθ, which is denoted by $\hat{\theta}_{BL}$ under the LINEX loss function, is the value which minimizes (6). It is given by,

$$\hat{\theta}_{BL} = -\frac{1}{\delta}\log\left(E_\theta\left[e^{-\delta\theta}|\underset{\sim}{x}\right]\right) \quad (7)$$

provided that the expectation $E_\theta\left[e^{-\delta\theta}|\underset{\sim}{x}\right]$ exists and is finite. The Bayes estimate of θ under SELF is the posterior mean of θ.

The main objective of this paper is to derive the estimates of the unknown parameters and reliability characteristics under Bayesian and non-Bayesian paradigm based on progressively type-II censoring scheme. These estimates are obtained for informative prior under two loss functions namely; squared error loss function and LINEX loss function receptively. It may be noted here that the estimates obtained are not in nice close forms and they can be analysed by any numerical integration technique. Here, we used MCMC technique to solve the integration involve in explicit equations. Also, we compare the MLEs with corresponding Bayes' estimators under the assumption of independent gamma prior of the unknown parameters by Monte-Carlo simulations. The rest of the paper is organized as follows: the maximum likelihood estimators (MLEs) of the parameters and reliability characteristics are obtained in section 2. In section 3, we have obtained Bayes estimators for unknown parameters of the (EIW) distribution under progressively type-II censoring scheme. The estimates are obtained based on the squared error loss function (SELF) and LINEX loss functions. The risk of estimates has been obtained. The comparison of Bayes estimator and correspond MLE under both loss functions in term of their simulated risks (*average loss over sample space*) have been studied in section 4. Finally, conclusions are presented in section 5.

2. Maximum Likelihood Estimation (MLE)

In this section, we discussed the maximum likelihood estimates (MLE's) of the parameters and reliability characteristics of exponential inverted Weibull (EIW) distribution. Suppose, we have n identical items put on test. Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ be progressive type-II censored samples of size m from a continuous distribution with distribution function (2) and density function (1) with $R_1, R_2, R_3, \ldots, R_m$ removals. For simplicity of notation, we will use x_i instead of $X_{i:m:n}$, with i=1,2,...,m. For progressive censoring with predetermined number of removals $R = (R_1 = r_1, R_2 = r_2, R_3 = r_3, \ldots, R_{m-1} = r_{m-1}, R_m = r_m)$. Then, the likelihood function based on all m progressively type-II censored samples is given by (see, Cohen [7] and Balakrishnan and Aggrawala [3]),

$$L_1(x;\theta,\beta/R=r) = C\prod_{i=1}^{m} f\,(x_i/\theta,\beta)[1-F(x_i/\theta,\beta)]^{R_i} \quad (8)$$

where C is a constant and expressed as,

$$C = n(n-R_1-1)(n-R_1-R_2-2)(n-R_1-R_2-R_3-3)\ldots(n-R_1-R_2-R_3-\ldots-R_{m-1}-m+1)$$

and $0 < R_i < (n-m-R_1-R_2-R_3-\ldots R_{i-1})$ for i=1,2,3,...,m-1.

Now, substituting (1) and (2) in equation (8), we get

$$L_1(x;\theta,\beta/R=r) = C\theta^m\beta^m e^{-\theta\sum_{i=1}^{m} x_i^{-\beta}} \prod_{i=1}^{m}\left(x_i^{-(\beta+1)}\right)\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i} \quad (9)$$

Here, it may be noted that the removals at each stage of the experiment is not fix i.e, R_i is a random variable and assume to be follows a Binomial law with specified probability p. Therefore, the probability of R_i removals after the i^{th} failure occurs, that is,

$$P(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \quad (10)$$

and for i = 2, 3,..., m-1,

$$P(R_i = r_i/R_{i-1} = r_{i-1}\ldots R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1}}{r_i} p^{r_i}(1-p)^{n-m-\sum_{j=1}^{i-1} r_j} \quad (11)$$

Now, further we assume that the removals R_i occurs is independent of $X_i's$ for all i. Then, the joint likelihood function of all $X = (X_1, X_2, X_3, \ldots, X_m)$ and $R = (R_1, R_2, R_3, \ldots, R_m)$ may be written as,

$$L(x, r; \theta, \beta, p) = L_1(x; \theta, \beta/R = r)P(R = r, p) \quad (12)$$

Where,

$$\begin{aligned} P(R = r; p) &= P(R_{m-1} = r_{m-1}/R_{m-2} = r_{m-2}, \ldots, R_1 = r_1) \\ &\quad \times (R_{m-2} = r_{m-2}/R_{m-3} = r_{m-3}, \ldots, R_1 = r_1) \\ &\quad \ldots \times P(R_2 = r_2/R_1 = r_1) \\ &= \frac{(n-m)!}{\left(n-m-\sum_{j=1}^{i-1} r_j\right)! \prod_{j=1}^{m-1} r_j} p^{\sum_{j=1}^{m-1} r_j}(1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \end{aligned} \quad (13)$$

Now, using (9), (12) and (13), we can write the combined likelihood function in the following form,

$$L(x, r; \theta, \beta, p) \propto \theta^m \beta^m e^{-\theta \sum_{i=1}^m x_i^{-\beta}} \prod_{i=1}^m \left(x_i^{-(\beta+1)}\right) \prod_{i=1}^m \left[1 - e^{-\theta x_i^{-\beta}}\right]^{R^i} p^{\sum_{j=1}^{m-1} r_j}(1 - p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \quad (14)$$

Then, log-likelihood (LogL=l) function can be written as,

$$\begin{aligned} l &= logL \propto m\ln\theta + m\ln\beta - (\beta+1)\sum_{i=1}^m \ln x_i - \theta \sum_{i=1}^m x_i^{-\beta} + R_i \sum_{i=1}^m \ln\left[1 - \left(e^{-x^{-\beta}}\right)^\theta\right] \\ &\quad + \sum_{j=1}^{m-1} r_j \ln p + (m-1)(n-m) - \sum_{j=1}^{m-1}(m-j)\, r_j \ln(1-p) \end{aligned} \quad (15)$$

The MLEs of θ and β can be found by simultaneously solving the following non-linear normal equations which are as follows,

$$\frac{\partial l}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^m x_i^{-\beta} + \sum_{i=1}^m R_i \frac{x_i^{-\beta} e^{-\theta x_i^{-\beta}}}{\left(1-e^{-\theta x_i^{-\beta}}\right)} \quad (16)$$

and

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^m \ln x_i + \theta\beta \sum_{i=1}^m R_i \frac{x_i^{-(\beta+1)} e^{-\theta x_i^{-\beta}}}{\left(1-e^{-\theta x_i^{-\beta}}\right)} \quad (17)$$

As we can see that the equations (16) and (17) are not in nice closed form, therefore we propose to use the N-R method to obtain the MLE's. If $\hat{\theta}$ and $\hat{\beta}$ are the MLE's of the parameters, then by using the invariance property of the MLE, the corresponding MLEs of the reliability $\hat{R}_{ML}(t)$ and hazard function $\hat{H}_{ML}(t)$ can be obtained by replacing θ by their MLEs in (3) and (4), respectively,

$$\hat{R}_{ML}(t) = 1 - \left(e^{-t^{-\hat{\beta}}}\right)^{\hat{\theta}} \quad (18)$$

$$\hat{H}_{ML}(t) = \frac{\hat{\theta}\hat{\beta} t^{-(\hat{\beta}+1)}\left(e^{-t^{-\hat{\beta}}}\right)^{\hat{\theta}}}{1-\left(e^{-t^{-\hat{\beta}}}\right)^{\hat{\theta}}} \quad (19)$$

Since, $P(R, p)$ does not depend on model parameters θ and β. Hence, the MLE of Binomial parameter p can be found by maximizing the likelihood function (15) directly. The first order derivative of equation (15) with respect to p is,

$$\frac{\partial L}{\partial p} = \frac{\sum_{j=1}^{m-1} r_j}{p} - \frac{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}{(1-p)} \quad (20)$$

putting $\left(\frac{\partial L}{\partial p}\right) = 0$ and solving, we get the MLE of p as,

$$\hat{p} = \frac{\sum_{j=1}^{m-1} r_j}{\sum_{j=1}^{m-1} r_j + \sum_{j=1}^{m-1}(m-j)r_j} \quad (21)$$

3. Bayes Estimation

In this section, we have developed the Bayesian estimation procedure for the parameterθ, reliability function and hazard function for the considered model based progressively type-II censored data with Binomial removals under both symmetric asymmetric loss functions namely, squared error and LINEX loss function. In order to obtain Bayes estimators of parameters θ and β, we must assume that the model parameters θ and β are randomly distributed. Thus, we need to specify a appropriate prior distribution for parameters. It may be noted that there does not exists any conjugate priors for unknown parameters θ and β consequently, we assume it as independently distributed. In such a case, there are many ways to choose the appropriate priors. Here, we considered independent gamma priors as our appropriate prior with density functions of the following forms for the unknown parameters θ and β,

$$g_1(\theta) \propto \theta^{a-1}e^{-b\theta}; \qquad \theta > 0, \qquad a, b > 0 \tag{22}$$

$$g_2(\beta) \propto \beta^{c-1}e^{-d\beta}; \qquad \beta > 0, \qquad c, d > 0 \tag{23}$$

where a, b, c and d are the hyper-parameters. Here, one can easily see that it covers wide variety of prior believes due to its high flexibility in nature and can be considered as suitable prior for θ and β. Thus, the joint prior distribution of (θ, β) as,

$$g(\theta, \beta) = g_1(\theta)g_2(\beta); \qquad \theta > 0, \beta > 0 \tag{24}$$

Then, combining the priors given by (24) with the likelihood function given by (14), we can obtain the joint posterior distribution of (θ, β) as,

$$\pi\left(\theta, \beta / \underset{\sim}{x}, r\right) \propto \theta^{m+a-1}\beta^{m+c-1}e^{-\left(b\theta+d\beta+\theta\sum_{i=1}^{m}x_i^{-\beta}\right)}\prod_{i=1}^{m}x_i^{-(\beta+1)}\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i} \tag{25}$$

Bayes estimator under squared error loss function$(\boldsymbol{SELF})$

from (25), the Bayes estimates of θ and β under squared error loss function can be derived as,

$$\hat{\theta}_{BS} = E_\pi\left[\theta|\underset{\sim}{x}, r\right] = k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}\theta^{m+a}\,\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta \tag{26}$$

and

$$\hat{\beta}_{BS} = E_\pi\left[\beta|\underset{\sim}{x}, r\right] = k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}\theta^{m+a-1}\,\beta^{m+c}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta \tag{27}$$

respectively. Now, Bayes estimates of the reliability and hazard function based on SELF may be obtained as,

$$\hat{R}(t)_{BS} = E_\pi\left[R(t)|\underset{\sim}{x}, r\right] = k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}\left(1-e^{-\theta t^{-\beta}}\right)\theta^{m+a-1}\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta \tag{28}$$

and

$$\hat{H}(t)_{BS} = E_\pi\left[H(t)|\underset{\sim}{x}, r\right] = k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}\frac{\theta^{m+a}\beta^{m+c}t^{-(\beta+1)}e^{-\theta t^{-\beta}}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)}{\left(1-e^{-\theta t^{-\beta}}\right)}d\theta d\beta \tag{29}$$

respectively.

Bayes estimator under Linex Loss Function$(\boldsymbol{LLF})$

The Bayes estimates of θ and β under Linex loss function (5) is written as,

$$\hat{\theta}_{BL} = -\frac{1}{\delta}\ln E_\pi\left[e^{-\delta\theta}|\underset{\sim}{x}, r\right] = -\frac{1}{\delta}\ln\left\{k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}e^{-\delta\theta}\,\theta^{m+a-1}\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta\right\} \tag{30}$$

and

$$\hat{\beta}_{BL} = -\frac{1}{\delta}\ln E_\pi\left[e^{-\delta\beta}|\underset{\sim}{x}, r\right] = -\frac{1}{\delta}\ln\left\{k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}e^{-\delta\beta}\,\theta^{m+a-1}\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta\right\} \tag{31}$$

respectively. Now, Bayes estimates of the reliability and hazard function based on Linex loss function can be describe as,

$$\begin{aligned}\widehat{R(t)}_{BL} &= -\frac{1}{\delta}\ln E_\pi\left[e^{-\delta R(t)}|\underset{\sim}{x}, r\right] \\ &= -\frac{1}{\delta}\ln\left\{k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}e^{-\delta\left(1-e^{-\theta t^{-\beta}}\right)}\theta^{m+a-1}\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta\right\}\end{aligned} \tag{32}$$

$$\hat{H}(t)_{BL} = -\frac{1}{\delta}\ln E_{\pi}\left[e^{-\delta H(t)}|\underset{\sim}{x},r\right]$$

and

$$= -\frac{1}{\delta}\ln\left\{k^{-1}\int_{\theta=0}^{\infty}\int_{\beta=0}^{\infty}\left[e^{-\frac{\delta\theta\beta t^{-(\beta+1)}e^{-\theta t^{-\beta}}}{\left(1-e^{-\theta t^{-\beta}}\right)}}\right]\theta^{m+a-1}\beta^{m+c-1}\phi(\theta,\beta)\zeta(\theta,\beta)\psi(\theta,\beta)d\theta d\beta\right\} \tag{33}$$

respectively.

Where,

$k = \theta^{m+a-1}\beta^{m+c-1}e^{-\left(b\theta+d\beta+\theta\sum_{i=1}^{m}x_i^{-\beta}\right)}\prod_{i=1}^{m}x_i^{-(\beta+1)}\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i}$,

$\phi = \phi\left(e^{-\left(b\theta+d\beta+\theta\sum_{i=1}^{m}x_i^{-\beta}\right)}\right)$,

$\psi = \psi\left(\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i}\right)$

and $\zeta = \zeta\left(\prod_{i=1}^{m}x_i^{-(\beta+1)}\right)$

As we can observe from the above expressions that the Bayes estimators obtained by using different loss function can not be solved analytically since it is not in nice close form. Therefore, one needs to use any numerical techniques to solve it. Here, we discuss the use of Gibbs sampling procedure (MCMC) to simulate samples from posterior distribution. For more details about this procedure, see Hastings [8] and Smith and Robert [14]. In MCMC technique, we have considered the Metropolis-Hastings algorithms to generate samples from posterior distributions. The Gibbbs sampling procedure is an algorithms for simulating from the full conditional posterior distribution while metropolis-hastings generates samples from an arbitrary proposal distribution (see, Metropolis et al. [10]). In order to apply this technique, the full conditional posterior distributions of the parameters θ and β can be written as,

$$\pi_1\left(\theta|\beta,\underset{\sim}{x},r\right) \propto \theta^{m+a-1}e^{-\theta\left(b+\sum_{i=1}^{m}x_i^{-\beta}\right)}\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i} \tag{34}$$

$$\pi_2\left(\beta|\theta,\underset{\sim}{x},r\right) \propto \beta^{m+c-1}e^{-\left(d\beta+\theta\sum_{i=1}^{m}x_i^{-\beta}\right)}\prod_{i=1}^{m}x_i^{-(\beta+1)}\prod_{i=1}^{m}\left[1-e^{-\theta x_i^{-\beta}}\right]^{R_i} \tag{35}$$

respectively. The following MCMC procedure taken into account to generate posterior samples from the above full conditionals.

- The initial guess of parameters θ and β, say (θ_0,β_0).
- Start with i=1.
- Using Metropolis-Hasting algorithm, generate posterior samples (θ_i,β_i) for θ and β from $\pi_1\left(\theta|\beta,\underset{\sim}{x},r\right)$ and $\pi_2\left(\beta|\theta,\underset{\sim}{x},r\right)$ respectively, where asymptotic normal distribution of full conditional densities are considered as proposal distributions.
- Repeat steps (II)-(III), for all i=1, 2, 3,...,N and obtained (θ_1,β_1), (θ_2,β_2),(θ_3,β_3),...,(θ_N,β_N).
- Now, on the basis of obtained sample in step (IV), compute the Bayes estimates of the parameters θ and β under different loss functions as,

Under SELF:

$\hat{\theta}_{BS} = E[\theta|data] \approx \frac{1}{N}\sum_{i=1}^{N}\theta_i$.

$\hat{\beta}_{BS} = E[\beta|data] \approx \frac{1}{N}\sum_{i=1}^{N}\beta_i$.

$\widehat{R(t)}_{BS} = E[R(t)|data] \approx \frac{1}{N}\sum_{i=1}^{N}\left(1-e^{-\theta_i t^{-\beta_i}}\right)$.

$\widehat{H(t)}_{BS} = E[H(t)|data] \approx \frac{1}{N}\sum_{i=1}^{N}\frac{\theta_i\beta_i t^{-(\beta_i+1)}\left(e^{-\theta_i t^{-\beta_i}}\right)}{1-\left(e^{-\theta_i t^{-\beta_i}}\right)}$.

Under Linex loss function:

$\hat{\theta}_{BL} = -\frac{1}{\delta}\ln\{E[\theta|data]\} \approx -\frac{1}{\delta}\ln\left\{\frac{1}{N}\sum_{i=1}^{N}e^{-\delta\theta_i}\right\}$.

$$\hat{\beta}_{BL} = -\frac{1}{\delta}\ln\{E[\beta|data]\} \approx -\frac{1}{\delta}\ln\left\{\frac{1}{N}\sum_{i=1}^{N} e^{-\delta\beta_i}\right\}.$$
$$\widehat{R(t)}_{BL} = -\frac{1}{\delta}\ln\{E[R(t)|data]\} \approx -\frac{1}{\delta}\ln\left\{\frac{1}{N}\sum_{i=1}^{N}\left(1-e^{-\theta_i t^{-\beta_i}}\right)\right\}.$$
$$\widehat{H(t)}_{BL} = -\frac{1}{\delta}\ln\{E[H(t)|data]\} \approx -\frac{1}{\delta}\ln\left\{\frac{1}{N}\sum_{i=1}^{N}\frac{\theta_i\beta_i t^{-(\beta_i+1)}\left(e^{-\theta_i t^{-\beta_i}}\right)}{1-\left(e^{-\theta_i t^{-\beta_i}}\right)}\right\}.$$

4. Simulation Study

In this section, we discuss the comparison study of the performances of the proposed estimators under SELF and LINEX loss functions with their corresponding maximum likelihood estimators (MLEs). The comparisons are based on the simulated risks (average loss over sample space). It may be mentioned here that the obtained MLEs and Bayes estimators under different loss functions are not in closed form. Therefore, N-R method and MCMC method have been used to compute these. These calculations are made on the basis of Monte Carlo simulation study of 1000 samples. Since, our study comprises the progressively type-II censored sample with Binomial removals, we need to simulate progressively type-II censored samples with Binomial removals from specified EIW distribution and propose to use of the following algorithms.

- Specify the value of n and m.
- Specify the value of parameters θ, β and p.
- Generate random number R_i using the Binomial law with specified probability i.e., $R_i \sim B\left(n-m-\sum_{j=1}^{i-1} R_j, p\right)$, for i=1, 2, 3,...,m-1.
- Set R_m according to the following relation: $R_m = \begin{cases} n-m-\sum_{j=1}^{m-1} R_j; & n-m-\sum_{j=1}^{m-1} R_j > 0 \\ 0; & otherwise. \end{cases}$
- Generate m independent U(0,1) random variables $Z_1, Z_2, Z_3, \dots, Z_m$.
- For the given values of the progressive type-II censoring schemes R_i (i=1, 2, 3,...,m). set $V_i = Z_i^{1/(i+R_m+\dots+R_{m-i+1})}$ ((i= 1, 2,...,m).
- Set $U_i = 1 - V_m V_{m-1}, \dots, V_{m-i+1}$ (i=1,2,...,m), then $U_1, U_2, U_3, \dots, U_m$ are the progressively type-II censored samples with Binomial removals of size m from U(0,1).
- Finally, for given values of parameters θ and β, set $x_i = F^{-1}(U)$ (i=1,2,3,...,m). Then, $(x_1, x_2, \dots, x_m)$ is the required progressive type-II censored samples with Binomial removals of size m from the EIWD. Here, we generate random sample of size 20, 30 and 50 from the specific distribution for fixed value of $p = 0.4$. The values of m are chosen such that the sample observations are 60% and 80% censored.
- Next, the MLEs and Bayes estimators of the parameters θ and β are evaluated. Also, the estimators of reliability R(t) and hazard functions H(t) may be obtained for specified time t=2 hrs. respectively.
- The Bayes estimates are obtained under gamma informative prior using squared error loss function and linex loss function.
- The choice of the hyper parameters may be chosen in such a way that if we consider any two independent information as prior mean and variance of θ and β then, the prior mean (say, ν) equals to the true value of the parameter with varying prior variance (say, η). The prior variance indicates the confidence of our prior guess. A large prior variance shows less confidence in prior guess and resulting prior distribution is relatively flat. On other hand, small prior variance indicates greater confidence in prior guess. In our case, we have chosen the prior mean equals to the true value of the parameter as ν is taken as 1.5 and 3.0 for θ and β, respectively with a prior variance η= 0.5 (small) giving ($a = 3.0, b = 4.5, c = 18, d = 6$).
- In order to consider linex loss function, we have taken four choices of loss parameter δ say, ($\delta = 0.5, 2.0$ for over estimation and $\delta = -0.5, -2.0$ for under estimation).
- Repeat the above steps 1000 times. Then we obtain the means and mean squared error (MSEs) for different sample size n and censoring sizes m. The ML estimates of θ, β, $R(t)$ and $H(t)$ and corresponding Bayes estimates for informative prior are obtained and are summarized in table(1-7).

- The estimators $\hat{\theta}_{ML}$ and $\hat{\beta}_{ML}$ represents the maximum likelihood estimators of the parameters θ and β respectively, while $\hat{\theta}_{BS}$, $\hat{\beta}_{BS}$ and $\hat{\theta}_{BL}$, $\hat{\beta}_{BL}$ are corresponding the Bayes estimators of parameters under SELF and Linex loss functions respectively.

From this extensive study, we may observed that the performances of Bayes estimator of θ, β, $R(t)$ and $H(t)$ better than the corresponding MLEs for all considered choices of the censoring parameters. Under linex loss function, we noticed that the risk of the Bayes estimators for $\delta = 0.5$ and $\delta = 2$ is least as compared to the negative values of the delta. We can also conclude that the maximum likelihood estimator is less efficient than the Bayes estimator since MLE shows the larger risk in all cases. All computational algorithms are calculated with help of R software.

Table 1: Estimate of θ and their corresponding MSE (in second rows) when p=0.4.

n	m	Criteria	$\hat{\theta}_{ML}$	$\hat{\theta}_{BS}$	δ=-0.5	δ=0.5	δ=-2	δ=2
					$\hat{\theta}_{BL1}$	$\hat{\theta}_{BL2}$	$\hat{\theta}_{BL3}$	$\hat{\theta}_{BL4}$
20	12	AE	1.6040	1.5491	1.5680	1.5307	1.6295	1.4791
		MSE	0.2297	0.1360	0.1484	0.1255	0.2011	0.1030
	16	AE	1.6213	1.5645	1.5824	1.5473	1.6424	1.4990
		MSE	0.2287	0.1306	0.1441	0.1193	0.2128	0.0950
30	18	AE	1.5615	1.5364	1.5467	1.5262	1.5783	1.4966
		MSE	0.1175	0.0925	0.0972	0.0882	0.1239	0.0777
	24	AE	1.5577	1.5323	1.5479	1.5172	1.5982	1.4744
		MSE	0.1162	0.0830	0.0900	0.0773	0.1218	0.0657
50	30	AE	1.5370	1.5256	1.5302	1.5210	1.5442	1.5073
		MSE	0.0728	0.0642	0.0660	0.0625	0.0719	0.0581
	40	AE	1.5457	1.5330	1.5410	1.5251	1.5656	1.5019
		MSE	0.0623	0.0535	0.0558	0.0514	0.0641	0.0462

Table 2: Estimate of β and their corresponding MSE (in second rows) when p=0.4.

n	m	Criteria	$\hat{\beta}_{ML}$	$\hat{\beta}_{BS}$	δ=-0.5	δ=0.5	δ=-2	δ=2
					$\hat{\beta}_{BL1}$	$\hat{\beta}_{BL2}$	$\hat{\beta}_{BL3}$	$\hat{\beta}_{BL4}$
20	12	AE	3.3744	3.1801	3.2126	3.1483	3.3135	3.0559
		MSE	0.8728	0.3323	0.3685	0.3001	0.5050	0.2253
	16	AE	3.3014	3.1872	3.2092	3.1654	3.2766	3.1012
		MSE	0.5711	0.2998	0.3232	0.2781	0.4050	0.2233
30	18	AE	3.2158	3.1463	3.1620	3.1306	3.2097	3.0843
		MSE	0.4455	0.2854	0.3009	0.2707	0.3533	0.2119
	24	AE	3.1999	3.1352	3.1524	3.1181	3.2048	3.0675
		MSE	0.3366	0.2138	0.2269	0.2016	0.2723	0.1704
50	30	AE	3.1211	3.1034	3.1089	3.0979	3.1255	3.0814
		MSE	0.2148	0.1865	0.1902	0.1828	0.2018	0.1725
	40	AE	3.1092	3.0907	3.0973	3.0841	3.1173	3.0643
		MSE	0.1679	0.1434	0.1469	0.1400	0.1581	0.1306

Table 3: Estimate of reliability function and their corresponding MSE (in second rows) when p=0.4.

n	m	Criteria	$\hat{R}_{ML}(t)$	$\hat{R}_{BS}(t)$	δ=-0.5	δ=0.5	δ=-2	δ=2
					$\hat{R}_{BL1}(t)$	$\hat{R}_{BL2}(t)$	$\hat{R}_{BL3}(t)$	$\hat{R}_{BL4}(t)$
20	12	AE	0.1615	0.1693	0.1695	0.1686	0.1708	0.1673
		MSE	0.2496	0.2413	0.2392	0.2401	0.2380	0.2413
	16	AE	0.1622	0.1683	0.1665	0.1658	0.1675	0.1649
		MSE	0.2466	0.2406	0.2410	0.2416	0.2401	0.2425
30	18	AE	0.1657	0.1702	0.1687	0.1683	0.1695	0.1676
		MSE	0.2429	0.2385	0.2390	0.2394	0.2383	0.2421
	24	AE	0.1630	0.1644	0.1668	0.1662	0.1677	0.1653
		MSE	0.2414	0.2330	0.2389	0.2384	0.2381	0.2413
50	30	AE	0.1675	0.1688	0.1682	0.1680	0.1685	0.1677
		MSE	0.2393	0.2322	0.2383	0.2384	0.2380	0.2388
	40	AE	0.1683	0.1699	0.1691	0.1688	0.1696	0.1684
		MSE	0.2379	0.2324	0.2368	0.2371	0.2364	0.2375

Table 4: Estimate of Hazard function and their corresponding MSE (in second rows) when p=0.4

n	m	Criteria	$\hat{H}_{ML}(t)$	$\hat{H}_{BS}(t)$	δ=-0.5 $\hat{H}_{BL1}(t)$	δ=0.5 $\hat{H}_{BL2}(t)$	δ=-2 $\hat{H}_{BL3}(t)$	δ=2 $\hat{H}_{BL4}(t)$
20	12	AE	6.2297	5.8215	5.9843	5.6789	6.4733	5.2562
		MSE	4.0952	2.2760	1.9942	1.3144	3.7144	0.8452
	16	AE	6.0734	5.8217	5.9483	5.7395	6.2733	5.4400
		MSE	2.6490	1.8307	1.6564	1.2113	2.6298	0.8071
30	18	AE	5.9018	5.7405	5.8356	5.6856	6.0648	5.4658
		MSE	2.1241	1.5892	1.5374	1.2362	2.1347	0.9213
	24	AE	5.8728	5.7618	5.8234	5.6580	6.0786	5.4174
		MSE	1.5871	1.2948	1.1542	0.9004	1.6970	0.6642
50	30	AE	5.7067	5.6662	5.6987	5.6454	5.7788	5.5656
		MSE	1.0309	1.0285	0.9351	0.8613	1.0608	0.7652
	40	AE	5.6781	5.6380	5.6746	5.6098	5.7721	5.5131
		MSE	0.8027	0.8553	0.7234	0.6545	0.8469	0.5707

5. Concluding remarks

In this paper, Bayesian and non-Bayesian estimation problems have been considered for the unknown parameters, reliability and hazard function of the exponentiated inverted Weibull distribution ($EIWD$) under the progressively type-II censoring scheme with Binomial removals. Since, the maximum likelihood estimator obtained is not in nice closed form hence, the N-R method is used to obtain the MLEs. Furthermore, we have considered Bayes estimation of the unknown parameters based on different loss functions and it's expressions cannot obtained in explicit form. Therefore, in order to solve this, MCMC technique has been utilised.

Finally, the Monte Carlo simulation study has been carried out to check the performances of the proposed estimators with their corresponding MLEs. On the basis of comparison of simulated risks of estimators, it is found that Bayes estimators perform better than maximum likelihood estimation and Bayes estimator under LINEX is also more efficient than the Bayes estimator under SELF in most of the situation since, we observed that the simulated risks of the Bayes estimators that are obtained based on LINEX loss function are smaller than the corresponding risks of the estimators, which are obtained, based on squared error loss function. So, Bayes estimate under Linex loss function is preferable to Bayes estimate under squared error loss function or asymmetric loss function is more appropriate than the symmetric loss function.

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